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Relaxation and Preconditioning for High Order Discontinuous Galerkin Methods with  
Applications to Aeroacoustics and High Speed Flows

Chi-Wang Shu

Brown University

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This project is about the investigation of the development of the discontinuous Galerkin finite element methods, for general geometry and triangulations, for solving convection dominated problems, with applications to aeroacoustics. Other related issues in high order WENO finite difference and finite volume methods have also been investigated.

High order WENO (weighted essentially non-oscillatory) schemes and discontinuous Galerkin methods are two classes of high order, high resolution methods suitable for convection dominated simulations with possible discontinuous or sharp gradient solutions. In [18], we first review these two classes of methods, pointing out their similarities and differences in algorithm formulation, theoretical properties, implementation issues, applicability, and relative advantages. We then present some quantitative comparisons of the third order finite volume WENO methods and discontinuous Galerkin methods for a series of test problems to assess their relative merits in accuracy and CPU timing.

In [3], we review the development of the Runge-Kutta discontinuous Galerkin (RKDG) methods for non-linear convection-dominated problems. These robust and accurate methods have made their way into the main stream of computational fluid dynamics and are quickly finding use in a wide variety of applications. They combine a special class of Runge-Kutta time discretizations, that allows the method to be non-linearly stable regardless of its accuracy, with a finite element space discretization by discontinuous approximations, that incorporates the ideas of *numerical fluxes* and *slope limiters* coined during the remarkable development of the *high-resolution* finite difference and finite volume schemes. The resulting RKDG methods are stable, high-order accurate, and highly parallelizable schemes that can easily handle complicated geometries and boundary conditions. We review the theoretical and algorithmic aspects of these methods and show several applications including nonlinear conservation laws, the compressible and incompressible Navier-Stokes equations, and Hamilton-Jacobi-like equations.

In [14], we develop a local discontinuous Galerkin method for solving KdV type equations containing third derivative terms in one and two space dimensions. The method is based on the framework of the discontinuous Galerkin method for conservation laws and the local discontinuous Galerkin method for viscous equations containing second derivatives, however the guiding principle for inter-cell fluxes and nonlinear stability is new. We prove  $L^2$  stability and a cell entropy inequality for the square entropy for a class of nonlinear PDEs of this type both in one and multiple spatial dimensions, and give an error estimate for the linear cases

in the one dimensional case. The stability result holds in the limit case when the coefficients to the third derivative terms vanish, hence the method is especially suitable for problems which are “convection dominate”, i.e. those with small second and third derivative terms. Numerical examples are shown to illustrate the capability of this method. The method has the usual advantage of local discontinuous Galerkin methods, namely it is extremely local and hence efficient for parallel implementations and easy for  $h$ - $p$  adaptivity.

In [15], we review the existing and develop new local discontinuous Galerkin methods for solving time dependent partial differential equations with higher order derivatives in one and multiple space dimensions. We review local discontinuous Galerkin methods for convection diffusion equations involving second derivatives and for KdV type equations involving third derivatives. We then develop new local discontinuous Galerkin methods for the time dependent bi-harmonic type equations involving fourth derivatives, and partial differential equations involving fifth derivatives. For these new methods we present correct interface numerical fluxes and prove  $L^2$  stability for general nonlinear problems. Preliminary numerical examples are shown to illustrate these methods. Finally, we present new results on a post-processing technique, originally designed for methods with good negative-order error estimates, on the local discontinuous Galerkin methods applied to equations with higher derivatives. Numerical experiments show that this technique works as well for the new higher derivative cases, in effectively doubling the rate of convergence with negligible additional computational cost, for linear as well as some nonlinear problems, with a local uniform mesh.

A virtual internal bond (VIB) model is proposed recently in mechanical engineering literatures for simulating dynamic fracture. The model is a nonlinear wave equation of mixed type (hyperbolic or elliptic). There is instability in the elliptic region and usual numerical methods might not work. In [9], we examine the artificial viscosity method for the model and apply central type schemes directly to the corresponding viscous system to ensure appropriate numerical viscous term for such a mixed type problem. We provide a formal justification of indicating convergence of the scheme despite the difficulty of the type change. The exact solution of a Riemann problem is used to demonstrate the numerical method for one dimensional case. We then generalize the method to a two dimensional material with a triangular or hexagonal lattice structure. Computational results for a two dimensional example are given.

In [2], we consider the enhancement of accuracy, by means of a simple post-processing technique, for finite element approximations to transient hyperbolic equations. The post-processing is a convolution with a kernel whose support has measure of order one in the case of arbitrary unstructured meshes; if the mesh is locally translation invariant, the support of the kernel is a cube whose edges are of size of the order of  $\Delta x$  only. For example, when polynomials of degree  $k$  are used in the discontinuous Galerkin (DG) method, and the exact solution is globally smooth, the DG method is of order  $k + 1/2$  in the  $L^2$  norm, whereas the post-processed approximation is of order  $2k + 1$ ; if the exact solution is in  $L^2$  only, in which case no order of convergence is available for the DG method, the post-processed approximation converges with order  $k + 1/2$  in  $L^2(\Omega_0)$  where  $\Omega_0$  is a subdomain over which

the exact solution is smooth. Numerical results displaying the sharpness of the estimates are presented.

In [17] we construct high order weighted essentially non-oscillatory (WENO) schemes for solving the nonlinear Hamilton-Jacobi equations on two-dimensional unstructured meshes. The main ideas are nodal based approximations, the usage of monotone Hamiltonians as building blocks on unstructured meshes, nonlinear weights using smooth indicators of second and higher derivatives, and a strategy to choose diversified smaller stencils to make up the bigger stencil in the WENO procedure. Both third-order and fourth-order WENO schemes using combinations of second-order approximations with nonlinear weights are constructed. Extensive numerical experiments are performed to demonstrate the stability and accuracy of the methods. High-order accuracy in smooth regions, good resolution of derivative singularities, and convergence to viscosity solutions are observed.

High order finite difference WENO methods have the advantage of simpler coding and smaller computational cost for multi-dimensional problems, compared with finite volume WENO methods of the same order of accuracy. However a main restriction is that conservative finite difference methods of third and higher order of accuracy can only be used on uniform rectangular or smooth curvilinear meshes. In order to overcome this difficulty, in [12] we develop a multi-domain high order WENO finite difference method which uses an interpolation procedure at the sub-domain interfaces. A simple Lagrange interpolation procedure is implemented and compared to a WENO interpolation procedure. Extensive numerical examples are shown to indicate the effectiveness of each procedure, including the measurement of conservation errors, orders of accuracy, essentially non-oscillatory properties at the domain interfaces, and robustness for problems containing strong shocks and complex geometry. Our numerical experiments have shown that the simple and efficient Lagrange interpolation suffices for the sub-domain interface treatment in the multi-domain WENO finite difference method, to retain essential conservation, full high order of accuracy, essentially non-oscillatory properties at the domain interfaces even for strong shocks, and robustness for problems containing strong shocks and complex geometry. The method developed in this paper can be used to solve problems in relatively complex geometry at a much smaller CPU cost than the finite volume version of the same method for the same accuracy. The method can also be used for high order finite difference ENO schemes and an example is given to demonstrate a similar result as that for the WENO schemes.

In [13] we address the issue of numerical resolution and efficiency of high order weighted essentially non-oscillatory (WENO) schemes for computing solutions containing both discontinuities and complex solution features, through two representative numerical examples: the double Mach reflection problem and the Rayleigh-Taylor instability problem. We conclude that for such solutions with both discontinuities and complex solution features, it is more economical in CPU time to use higher order WENO schemes to obtain comparable numerical resolution.

A quantitative study is carried out in [16] to investigate the size of numerical viscosities and the resolution power of high order WENO (weighted essentially non-oscillatory) schemes for solving one and two dimensional Navier-Stokes equations for compressible gas dynamics

with high Reynolds numbers. A one-dimensional shock tube problem, a one-dimensional example with parameters motivated by supernova and laser experiments, and a two dimensional Rayleigh-Taylor instability problem are used as numerical test problems. For the two dimensional Rayleigh-Taylor instability problem, or similar problems with small scale structures, the details of the small structures are determined by the physical viscosity (therefore, the Reynolds number) in the Navier-Stokes equations. Thus, to obtain faithful resolution to these small scale structures, the numerical viscosity inherent in the scheme must be small enough so that the physical viscosity dominates. A careful mesh refinement study is performed to capture the threshold mesh for full resolution, for specific Reynolds numbers, when WENO schemes of different orders of accuracy are used. It is demonstrated that high order WENO schemes are more CPU time efficient to reach the same resolution, both for the one dimensional and two dimensional test problems.

In [10] we further explore a local post-processing technique, originally developed by Bramble and Schatz using continuous finite element methods for elliptic problems and later by Cockburn, Luskin, Shu and Süli [2] using discontinuous Galerkin methods for hyperbolic equations. We investigate the technique in the context of superconvergence of the derivatives of the numerical solution, two space dimensions for both tensor product local basis and the usual  $k$ -th degree polynomials basis, multi-domain problems with different mesh sizes, variable coefficient linear problems including those with discontinuous coefficients, and linearized Euler equations applied to an aeroacoustic problem. We demonstrate through extensive numerical examples that the technique is very effective in all these situations in enhancing the accuracy of the discontinuous Galerkin solutions.

In [11] we study a class of one-sided post-processing techniques to enhance the accuracy of the discontinuous Galerkin methods. The applications considered in this paper are linear hyperbolic equations, however the technique can be used for the solution to a discontinuous Galerkin method solving other types of partial differential equations, or more general approximations, as long as there is a higher order negative norm error estimate for the numerical solution. The advantage of the one-sided post-processing is that it uses information only from one side, hence it can be applied up to domain boundaries, a discontinuity in the solution, or an interface of different mesh sizes. This technique allows us to obtain an improvement in the order of accuracy from  $k+1$  of the discontinuous Galerkin method to  $2k+1$  of the post-processed solution, using piecewise polynomials of degree  $k$ , throughout the entire domain and not just away from the boundaries, discontinuities, or interfaces of different mesh sizes.

The dynamics of a plane diode is described by the Vlasov-Poisson system over an interval with inflow boundary conditions at two ends. In [5], the uniqueness and regularity of such dynamics is investigated. It is shown that a rather general initial and boundary datum leads to a unique solution with bounded variations. Moreover, such a solution becomes discontinuous if the external voltage is large enough, while it can remain  $C^1$  if the external voltage is sufficiently small or absent.

In [1], we develop the locally divergence-free discontinuous Galerkin method for numerically solving the Maxwell equations. The distinctive feature of the method is the use of

approximate solutions that are exactly divergence-free inside each element. As a consequence, this method has a smaller computational cost than the one of the discontinuous Galerkin method with standard piecewise polynomial spaces. We show that, in spite of this fact, it produces approximations of the same accuracy. We also show that this method is more efficient than the discontinuous Galerkin method using globally divergence-free piecewise polynomial bases. Finally, a post-processing technique is used to recover  $(2k+1)$ -th order of accuracy when piecewise polynomials of degree  $k$  are used.

In [7], we continue our investigation of the locally divergence-free discontinuous Galerkin method, originally developed for the linear Maxwell equations in [1], to solve the nonlinear ideal magnetohydrodynamics (MHD) equations. The distinctive feature of such methods is the use of approximate solutions that are exactly divergence-free inside each element for the magnetic field. As a consequence, this method has a smaller computational cost than the traditional discontinuous Galerkin method with standard piecewise polynomial spaces. We formulate the divergence-free discontinuous Galerkin method for the MHD equations and perform extensive one and two dimensional numerical experiments for both smooth solutions and solutions with discontinuities. Our computational results demonstrate that the divergence-free discontinuous Galerkin method, with a reduced cost comparing to the traditional discontinuous Galerkin method, can maintain the same accuracy for smooth solutions and can enhance the numerical stability of the scheme and reduce certain nonphysical features in some of the test cases.

In [6], we develop local discontinuous Galerkin (DG) methods for solving nonlinear dispersive partial differential equations that have compactly supported traveling waves solutions, the so-called “compactons”. The schemes we present extend the previous works of Yan and Shu on approximating solutions for linear dispersive equations and for certain KdV-type equations. We present two classes of DG methods for approximating solutions of such PDEs. First, we generate nonlinearly-stable numerical schemes with a stability condition that is induced from a conservation law of the PDE. An alternative approach is based on constructing linearly-stable schemes, i.e. schemes that are linearly stable to small perturbations. The numerical simulations we present verify the desired properties of the methods including their expected order of accuracy. In particular, we demonstrate the potential advantages of using DG methods over pseudo-spectral methods in situations where discontinuous fronts and rapid oscillations co-exist in a solution.

In [8], we reinterpret a discontinuous Galerkin method originally developed by Hu and Shu for solving Hamilton-Jacobi equations. By this reinterpretation, numerical solutions will automatically satisfy the curl-free property of the exact solutions inside each element. This new reinterpretation allows a method of lines formulation, which renders a more natural framework for stability analysis. Moreover, this reinterpretation renders a significantly simplified implementation with reduced cost, as only a smaller subspace of the original solution space is used and the least square procedure used before is completely avoided.

There are 16 papers published or accepted in refereed journals and 2 preprints submitted in this period acknowledging support from this NASA grant. These papers are listed in the References.

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